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On the Bunge–Kalnay Position Operator for the Dirac Electron

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Abstract

A similarity is noted between a constant of the motion for the Dirac equation, with position operators as discussed by Bunge and Kalnay, and a constant of the motion discussed by Corben in connection with a nonrelativistic spinning particle.

The Dirac Hamiltonian, H, for a free particle of mass m is

$$H = c\boldsymbol{\alpha} \cdot \mathbf{P} + \beta mc$$

where the components of $\mathbf{P}$ are the momentum operators and β and the components of $\boldsymbol{\alpha}$ satisfy the usual anticommutation relations

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk} I, \qquad \{\alpha_j, \beta\} = 0, \qquad \{\beta, \beta\} = 2I$$

Following Bunge and Kalnay (1969), who discuss the properties of the position operators $\mathbf{X}$ defined by

$$\mathbf{X} = \mathbf{Q} + (i\hbar/2mc)\beta\boldsymbol{\alpha}$$

where the components of $\mathbf{Q}$ are the canonical position operators satisfying the commutation relations

$$[P_j, Q_k] = -i\hbar\delta_{jk} I, \qquad [\alpha_j, Q_k] = 0, \qquad [\beta, Q_k] = 0$$

we consider the family of position operators $\mathbf{X}^{(\lambda)}$ defined by

$$\mathbf{X}^{(\lambda)} = \mathbf{Q} + \lambda\beta\boldsymbol{\alpha}$$

The corresponding velocity operators $\mathbf{V}^{(\lambda)}$ and angular momentum operators $\mathbf{L}^{(\lambda)}$ are defined by

$$\mathbf{V}^{(\lambda)} = \frac{i}{\hbar}[H, \mathbf{X}^{(\lambda)}], \qquad \mathbf{L}^{(\lambda)} = \mathbf{X}^{(\lambda)} \wedge \mathbf{P}$$

We see that the angular momentum will be a constant of the motion if

$$\mathbf{V}^{(\lambda)} \wedge \mathbf{P} = \mathbf{0}$$

a condition which may be described by saying that the velocity and the momentum are parallel. This condition will hold if λ takes the Bunge-Kalnay value of $i\hbar/2mc$. This is in fact the only value of λ for which the velocity operators commute with one another.

We now consider the Dirac Hamiltonian, H', for a particle in a magnetic field described by the vector potential $\mathbf{A}$ which satisfies the transversality condition that $\nabla \cdot \mathbf{A} = 0$. We take H' to be given by

$$H' = c\boldsymbol{\alpha}\cdot \left[\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{X}^{(\lambda)})\right] + \beta mc^2$$

If we write $\mathbf{A}(\mathbf{X}^{(\lambda)}) = \mathbf{A}(\mathbf{Q} + \lambda\beta\boldsymbol{\alpha})$, expand the right-hand side as a Taylor series in λ, and ignore terms of order λ^2, we obtain

$$H' = c\boldsymbol{\alpha}\left[\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{Q})\right] + \beta\left(mc^2 - \frac{2ie\lambda}{\hbar}\,\mathbf{S}\cdot\mathbf{B}\right)$$

where $2S_1 = i\hbar\alpha_3\alpha_2$ etc., and $\mathbf{B} = \nabla \wedge \mathbf{A}$. Then

$$H'^2 = c^2\left[\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{Q})\right]^2 - 2ce\mathbf{S}\cdot\mathbf{B} + m^2c^4 - 4mc^2\,\frac{ie\lambda}{\hbar}\,\mathbf{S}\cdot\mathbf{B}$$

where a term of order λ^2 has again been ignored. Now if λ takes the Bunge-Kalnay value we obtain

$$H'^2 = c^2\left[\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{Q})\right]^2 + m^2c^4$$

We also note that for this value of λ we retain the result that the velocity is parallel to the momentum provided that we ignore terms of order λ^2.

Thus we obtain the result that when the velocity is parallel to the momentum $[\mathbf{P} - (e/c)\mathbf{A}(\mathbf{Q})]^2$ is a constant of the motion for the Hamiltonian H'. More importantly it is a constant of the motion that does not contain the operators $\mathbf{S}$, which are of course related to the spin of the particle. In this respect the result compares with a result due to Corben (1968), which asserts that where $\boldsymbol{\pi}$ is the mechanical momentum of a nonrelativistic particle moving in a constant magnetic field $\mathbf{B}$, $\boldsymbol{\pi}^2$ is a constant of the motion provided that two of the three vectors $\mathbf{V}, \boldsymbol{\pi}$, and $\mathbf{B}$ are parallel. If, on the other hand, no two of the three vectors $\mathbf{V}, \boldsymbol{\pi}$, and $\mathbf{B}$ are parallel then $\pi^2 - 2(e/c)\mathbf{S}\cdot\mathbf{B}$ is a constant of the motion. The similarity does not lie in any claim that $\mathbf{P} - (e/c)\mathbf{A}(\mathbf{Q})$ is the mechanical momentum, since the natural expression for the mechanical momentum is $\mathbf{P} - (e/c)\mathbf{A}(\mathbf{X}^{(\lambda)})$, but rather in the fact that when the velocity is parallel to the momentum the constants of the motion do not involve the operators $\mathbf{S}$.

References

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